PC5215 Numerical Recipes with Applications

Midterm test, 2 Oct 2025, closed book, 1:30 min

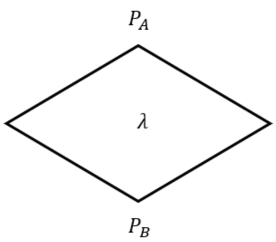
1. Give definitions or explain briefly for the following:
    a. The definition of norm for a vector
    b. Definition of the condition number of a square matrix, and why it is useful for linear equations Ax = b?
    c. Explain Gaussian quadrature
    d. Computational complexity of LU decomposition

a) The definition of norm is $\|v\| \geq 0$ $(0\ only\ for\ x\ = 0)$, $\|v + w\| \leq \|v\| + \|w\|$, $\|cv\| = |c|\|v\|$. *c is a scalar. b)* $\mathrm{cond}(A) = \|A\|\ \|A^{-1}\|$ *, useful for analyzing stability, or bound the error of solution. c) adjustable abscissa, exact for polynomials of degree 2N-1 for an N point formula* $\int_a^b f(x)dx = \sum_{i=1}^{N} w_i f(x_i)$*, determine* x_i*using the root of orthogonal polynomial* P_N*. d)* $O(N^3)$ *where the matrix size is* $N \times N$.

2. IEEE floating-point number of half precision contains 16 bits, with 1 highest bit for the sign, 5 bits for the biased exponent, and the remaining 10 bits for the fractional part. Based on this information, determine the following:
    a. the machine epsilon,
    b. the largest possible value,
    c. work out the bit pattern for the numerical value of $\pi \approx 3.141592653$. How many digits are exactly represented with half precision for π?

a) $\epsilon = 2^{-10} \approx 10^{-3}$ *b) 0 11110 1111111111 in bits which is* $2^{15}(2 - 2^{-10}) = 2^{16} - 32 = 65504$. *c) sign = 0, biased exponent e = 15 + 1 = 16, 3 -> 11, 0.14159… = 0.001001000011, so the IEEE bit pattern is 0 10000 1001001000 which is 3.140625 accurate to 3 digits.*

3. Consider Neville's algorithm of polynomial interpolation using the P table. This means the triangular table of P values can be updated according to



$P = \lambda P_A + (1 - \lambda)P_B$

Write Python code ptable(xar,yar, x, y) that takes two arrays for the given xar and yar values and returns the interpolated value at x as y. Your code is intended to run without error as written.

```
def Ptable(xar, yar, x, y):
   n = len(xar)
   P = yar.copy()
   for m in range(1,n):
     for i in range(0,n-m):
        lam = (x - xar[i+m])/(xar[i] – xar[i+m])
        P[i] = lam*P[i] + (1.0-lam)*P[i+1]
   y[0] = P[0]
```

4. Consider the open formula of integration where the first point $x_1 = 0$ with $f_1 = f(x_1)$ cannot be used, but the next two points $x_2 = h$ and $x_3 = 2h$ can be sampled to give f_2 and f_3, respectively. a). Determine the weights w_2 and w_3 and the error, i.e., the power to h:

$$\int_0^h f(x)dx = w_2 f_2 + w_3 f_3 + O(h^?)\,.$$

b). However, if the abscissa x_2 is allowed to adjust, but keeping x_3 fixed at $2h$, what will be the new formula and the accuracy?

a) Use the formula to try integrate 1 and x exactly, we get two equations, $h = w_2 + w_3$, $\frac{1}{2}h^2 = w_2 h + w_3(2h)$, solve for w we get $w_2 = \frac{3}{2}h$, $w_3 = -\frac{h}{2}$. We get error when we try $f(x) = x^2$, so the error is $O(h^3)$. b) This is not a gaussian quadrature as the last point $x_3 = 2h$ is not allowed to adjust. We now try polynomials $1, x, x^2$. The equations for 1 and x are the same as before except x_2 is a variable, i.e., $h = w_2 + w_3$, $\frac{1}{2}h^2 = w_2 x_2 + w_3(2h)$, and $\frac{1}{3}h^3 = w_2 x_2^2 + w_3(2h)^2$. Eliminate w_1, taking the ratio of the remaining two equations, we find $x_2 = \frac{4h}{9}$, $w_2 = \frac{27}{28}h$, $w_3 = \frac{h}{28}$, and the error is one order higher, i.e. $O(h^4)$.